Algebraic Number Theory
Exercise Sheet 2

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Winter Semester 2024-25
29.10.2024

Exercise 1. Let A be an integral domain and let F be the field of fractions of A. Let B be a ring, such that $A \subseteq B \subseteq F$. Show that the canonical morphism of fields $f : F \to K$ is an isomorphism, where K is the field of fractions of B.

Exercise 2. Let B be an integral domain and A a subring of B such that B is integral over A. Show that B is a field if and only if A is a field.

Exercise 3. Let A be an integrally closed ring. Let K be the field of fractions of A. Let $K \subset L$ be a finite field extension of K , B the integral closure of A in L and let $x \in B$. Show that x is a unit in B if and only if $N_K^L(x)$ is a unit in A.

Exercise 4. Let d be a square-free integer. Let $L = \mathbb{Q}(\alpha)$ be a quadratic field, where $\alpha^2 = d$. Let $\mathcal{O}_L$ be the ring of elements in L integral over $\mathbb{Z}$.

(0) Show that the $\mathbb{Q}$-linear map $\mathrm{Tr}_{\mathbb{Q}}^L : L \to \mathbb{Q}$ and the norm homomorphism $N : L^* \to \mathbb{Q}^*$ induce respectively a $\mathbb{Z}$-linear map $\mathrm{Tr} : \mathcal{O}_L \to \mathbb{Z}$ and a homomorphism $N : \mathcal{O}_L^* \to \{1, -1\}$.
(1) Describe the kernel and image of $\mathbb{Z}$-linear map $\mathrm{Tr} : \mathcal{O}_L \to \mathbb{Z}$.
(2) Consider the induced homomorphism $N : \mathcal{O}_L^* \to \{1, -1\}$. Show that
  (2.1) if $d < 0$, the homomorphism N is trivial.
  (2.2) if $d = 2$, the homomorphism N is surjective.
  (2.3) if $d = 3$, the homomorphism N is trivial.